§6.1 THE EXPONENTIAL FUNCTION & ITS INVERSE

* ACTIVITY *

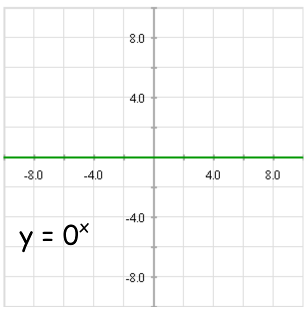
Exponents are a class of functions different from polynomials. With exponential functions, the exponent varies while the base remains the same. With polynomial functions, the base varies while the exponent remains constant.

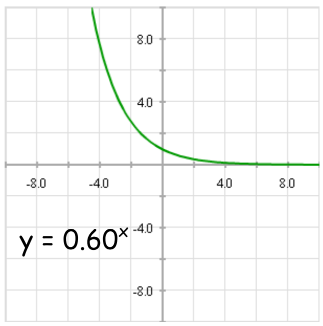
In this lesson, we will look at exponential functions of the form $\mathbf{y = Mb^{kx}}$, where **M**, **b**, and **k** are **constants**.

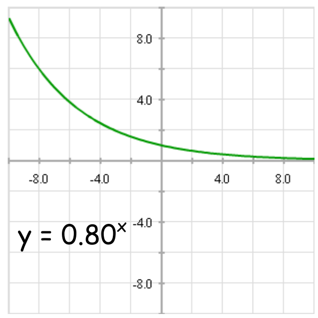
Below are several questions designed to get you thinking about exponential functions.

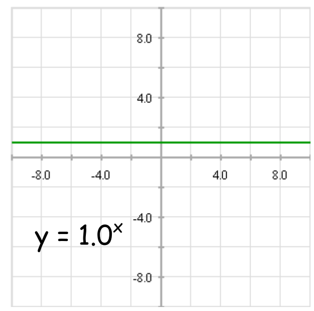
1. Set M = 1 and k = 1 to give us $y = b^x$.

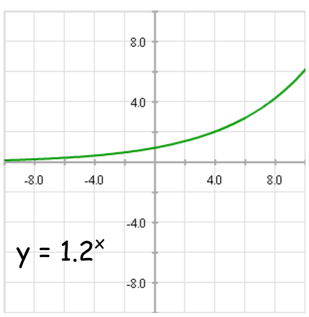
a) What effect does changing b have on the graph of the exponential?

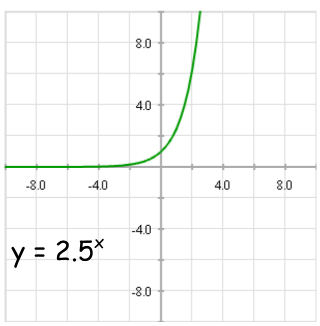


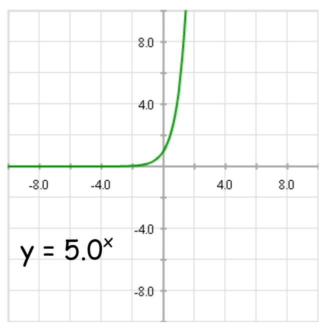


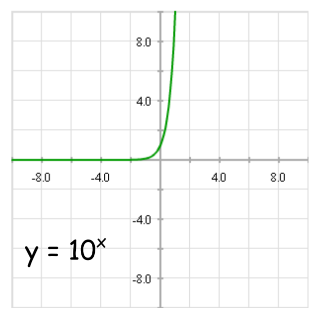












b) What is the y-intercept of the graph? Does it depend on the value of b?

c) What happens when b = 1? Explain.

d) When b is less than one, why does the graph decrease as x increases?

2. Construct a table of values for x = 1, 2, 3, 4, and 5, and compare the first differences for the functions $y = 3x$. $y = 3x^2$, and $y = 3^x$. Describe the rate of change of each.

$y = 3x$

x	y	1st Dif.
0		
1		
2		
3		
4		
5		

$y = 3x^2$

x	y	1st Dif.
0		
1		
2		
3		
4		
5		

$y = 3^x$

x	y	1st Dif.
0		
1		
2		
3		
4		
5		

- For the linear function $y = 3x$, the first differences have a ________________ value of _____. This ________________ rate of change, _____, is the slope of the line.
- For the quadratic function $y = 3x^2$, the first differences increase by _____ each time. They form an arithmetic sequence with the first term _____ and a common difference _____.
- For the exponential function $y = 3^x$, the first differences increase by a factor of _____ each time. They form a geometric sequence with the first term _____ and a common ratio _____.

Summary:

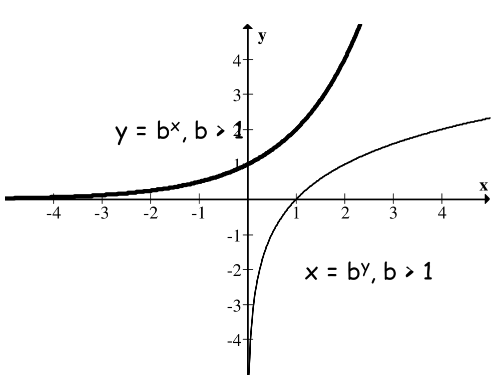
- Linear growth:
  - graph is a straight line
  - first differences are constant, the rate of change is constant
- Quadratic growth:
  - graph is a parabola
  - first differences form an arithmetic sequence (i.e. have a common difference)
- Exponential growth:
  - graph is an exponential curve
  - first differences form a geometric sequence (i.e. have a common ratio)

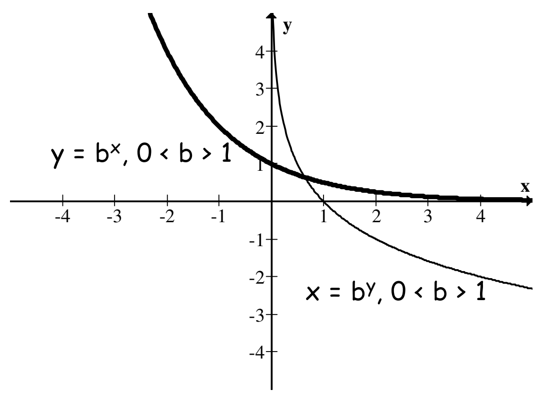
KEY CONCEPTS

- An exponential function can be written in the form of $f(x) = Mb^{kx}$, where M, b, and k are constants, and $b > 0$, $b \neq 1$, and has the following properties:
  - Domain: all real numbers (i.e. $x \in \Re$)
    Range: all positive real numbers (when $M > 0$)
  - For $b > 1$, $y = b^{kx}$ is an increasing function.
  - For $0 < b < 1$, $y = b^{kx}$ is a decreasing function.
  - First differences form a geometric sequence (i.e. have a common ratio)

Note: We will only be dealing with case where $M = 1$.

- The inverse of $y = b^x$ is a function that can be written as $x = b^y$. This function
  - has domain $\{ x \in \Re, x > 0 \}$
  - has range $\{ y \in \Re \}$
  - has x-intercept 1
  - has a vertical asymptote at $x = 0$
  - is a reflection of $y = b^x$ about the line $y = x$.



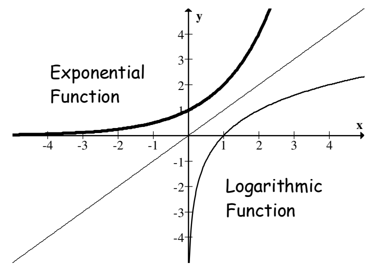


- is increasing on its domain when $b > 1$
- is decreasing on its domain when $0 < b < 1$

§6.2 LOGARITHMS

KEY CONCEPTS

- The logarithmic function is the inverse of the exponential function.



- The value of $\log_b x$ is equal to the exponent to which the base, b, is raised to produce x.

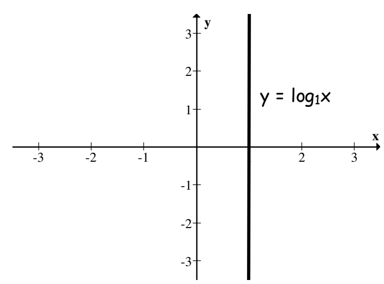
- Exponential equations can be written in logarithmic form, and vice versa.

  $y = b^x \quad \Longleftrightarrow \quad x = \log_b y$
  $y = \log_b x \quad \Longleftrightarrow \quad x = b^y$

- Exponential and logarithmic functions are defined only for positive values of the base that are not equal to one:

  $y = b^x$, $b > 0$, $x > 0$, $b \neq 1$
  $y = \log_b x$, $b > 0$, $y > 0$, $b \neq 1$

  The logarithm of x to base 1 is only valid when x = 1, in which case y has an infinite number of solutions and is not a function.



- Common logarithms are logarithms with base 10. It is not necessary to write the base for these logarithms: log x means the same as $\log_{10} x$.

Examples:

1. Rewrite the equation $6561 = 3^8$ in logarithmic form.

2. Evaluate each of the following.

a) $\log_4 1024$ **b)** $\log_3\left(\frac{1}{81}\right)$

3. Rewrite each of the following in exponential form.

a) $\log_5 125 = 3$ **b)** $y = \log 2x$

4. Find an approximate value for $\log_3 20$.

§6.3 TRANSFORMATIONS OF LOGARITHMIC FUNCTIONS

KEY CONCEPTS

- The techniques for applying transformations to logarithmic functions are the same as those for other functions.
  - $y = \log x + c$ — Translate up (if $c > 0$) or down (if $c < 0$) c units.
  - $y = \log(x - d)$ — Translate right (if $d > 0$) or left (if $d < 0$) d units.
  - $y = a \log x$ — Vertical stretch by a factor of $|a|$ if $|a| > 1$.
    Vertical compression by a factor of $|a|$ if $|a| < 1$.
    Reflect in the x-axis if $a < 0$.
  - $y = \log(kx)$ — Horizontal compression by a factor of $\left|\frac{1}{k}\right|$ if $|k| > 1$.
    Horizontal stretch by a factor of $\left|\frac{1}{k}\right|$ if $|k| < 1$, $k \neq 0$.
    Reflect in the y-axis if $k < 0$.
- It is easier to perform multiple transformations in a series of steps:
  1. Ensure that the function is in the form $f(x) = a \log[k(x - d)] + c$.
  2. Apply any horizontal or vertical stretches or compressions.
  3. Apply any reflections.
  4. Apply any horizontal or vertical translations.
- **Note:** When applying multiple transformations, it can be helpful to focus on certain anchor points, such as (1, 0) and (10, 1), as well as the position of the asymptote.

Examples:

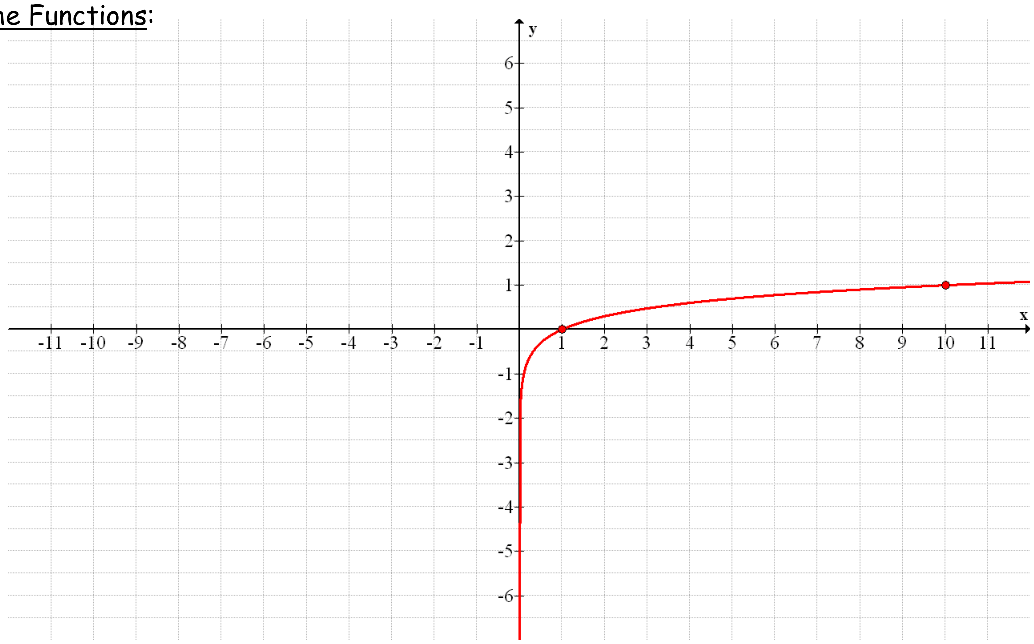
1. Sketch the graph of each function.

   a) $f(x) = 3\log[2(x + 4)] - 2$

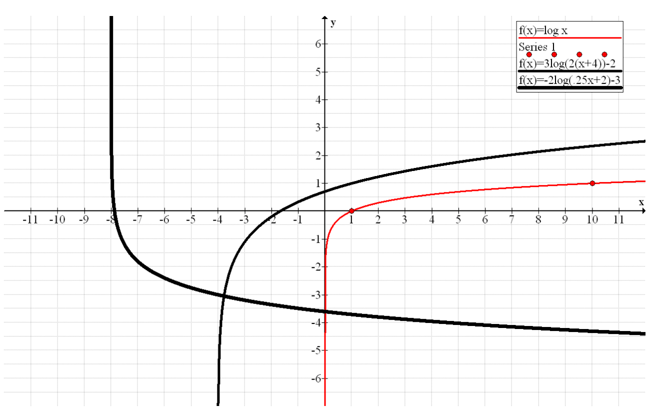
   b) $y = -2\log\left(\frac{1}{4}x + 2\right) - 3$

Work:

Graph the Functions:



Verify Final Graphs:



§6.4 POWER LAW OF LOGARITHMS

KEY CONCEPTS

- The **power law** of logarithms states that $\log_b x^n = n \log_b x$ **for** $b > 0$, $b \neq 1$, $x > 0$, **and** $n \in \Re$. This property can be used to solve equations involving logs.

- Any logarithm can be expressed in terms of common logarithms using the **change of base formula**: $\log_b m = \dfrac{\log m}{\log b}$, $b > 0$, $b \neq 1$, $m > 0$

  This formula can be used to evaluate logarithms or graph logarithmic functions with any base.

Examples:

2. Solve $5^x = 212$.

3. Evaluate $\log_4 20$, correct to 3 decimal places.

4. Amy invested $5,600 in a CD that pays eight percent compounded quarterly. The amount, A, that the investment is worth, as a function of time, t, in years, is given by $A(t) = 5600(1.02)^{4t}$.

a) How much will Amy's investment be worth at the end of 5 years?

b) How long will it take for Amy's investment to triple?